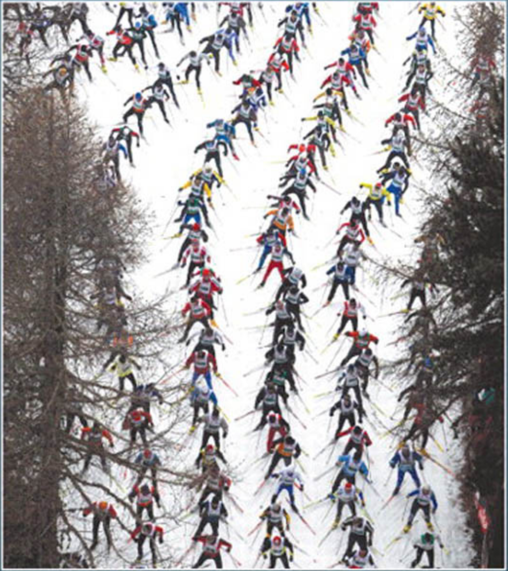
Vector Calculus

Fourth Edition



Susan Jane Colley

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Vector Calculus

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Vector Calculus 4th EDITION

Susan Jane Colley
Oberlin College

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To *Will and Diane*,
with love

About the Author
Susan Jane Colley

John Seyfried

Susan Colley is the Andrew and Pauline Delaney Professor of Mathematics at Oberlin College and currently Chair of the Department, having also previously served as Chair.

She received S.B. and Ph.D. degrees in mathematics from the Massachusetts Institute of Technology prior to joining the faculty at Oberlin in 1983.

Her research focuses on enumerative problems in algebraic geometry, particularly concerning multiple-point singularities and higher-order contact of plane curves.

Professor Colley has published papers on algebraic geometry and commutative algebra, as well as articles on other mathematical subjects. She has lectured internationally on her research and has taught a wide range of subjects in undergraduate mathematics.

Professor Colley is a member of several professional and honorary societies, including the American Mathematical Society, the Mathematical Association of America, Phi Beta Kappa, and Sigma Xi.

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Preface

Physical and natural phenomena depend on a complex array of factors. The sociologist or psychologist who studies group behavior, the economist who endeavors to understand the vagaries of a nation's employment cycles, the physicist who observes the trajectory of a particle or planet, or indeed anyone who seeks to understand geometry in two, three, or more dimensions recognizes the need to analyze changing quantities that depend on more than a single variable. Vector calculus is the essential mathematical tool for such analysis. Moreover, it is an exciting and beautiful subject in its own right, a true adventure in many dimensions.

The only technical prerequisite for this text, which is intended for a sophomore-level course in multivariable calculus, is a standard course in the calculus of functions of one variable. In particular, the necessary matrix arithmetic and algebra (not linear algebra) are developed as needed. Although the mathematical background assumed is not exceptional, the reader will still be challenged in places.

My own objectives in writing the book are simple ones: to develop in students a sound conceptual grasp of vector calculus and to help them begin the transition from first-year calculus to more advanced technical mathematics. I maintain that the first goal can be met, at least in part, through the use of vector and matrix notation, so that many results, especially those of differential calculus, can be stated with reasonable levels of clarity and generality. Properly described, results in the calculus of several variables can look quite similar to those of the calculus of one variable. Reasoning by analogy will thus be an important pedagogical tool. I also believe that a conceptual understanding of mathematics can be obtained through the development of a good geometric intuition. Although I state many results in the case of n variables (where n is arbitrary), I recognize that the most important and motivational examples usually arise for functions of two and three variables, so these concrete and visual situations are emphasized to explicate the general theory. Vector calculus is in many ways an ideal subject for students to begin exploration of the interrelations among analysis, geometry, and matrix algebra.

Multivariable calculus, for many students, represents the beginning of significant mathematical maturation. Consequently, I have written a rather expansive text so that they can see that there is a story behind the results, techniques, and examples—that the subject coheres and that this coherence is important for problem solving. To indicate some of the power of the methods introduced, a number of topics, not always discussed very fully in a first multivariable calculus course, are treated here in some detail:

- an early introduction of cylindrical and spherical coordinates (§1.7);
- the use of vector techniques to derive Kepler's laws of planetary motion (§3.1);
- the elementary differential geometry of curves in $\mathbf{R}^3$, including discussion of curvature, torsion, and the Frenet–Serret formulas for the moving frame (§3.2);
- Taylor's formula for functions of several variables (§4.1);

- the use of the Hessian matrix to determine the nature (as local extrema) of critical points of functions of n variables (§4.2 and §4.3);
- an extended discussion of the change of variables formula in double and triple integrals (§5.5);
- applications of vector analysis to physics (§7.4);
- an introduction to differential forms and the generalized Stokes's theorem (Chapter 8).

Included are a number of proofs of important results. The more technical proofs are collected as addenda at the ends of the appropriate sections so as not to disrupt the main conceptual flow and to allow for greater flexibility of use by the instructor and student. Nonetheless, some proofs (or sketches of proofs) embody such central ideas that they are included in the main body of the text.

New in the Fourth Edition

I have retained the overall structure and tone of prior editions. New features in this edition include the following:

- 210 additional exercises, at all levels;
- a new, optional section (§5.7) on numerical methods for approximating multiple integrals;
- reorganization of the material on Newton's method for approximating solutions to systems of n equations in n unknowns to its own (optional) section (§2.7);
- new proofs in Chapter 2 of limit properties (in §2.2) and of the general multivariable chain rule (Theorem 5.3 in §2.5);
- proofs of both single-variable and multivariable versions of Taylor's theorem in §4.1;
- various additional refinements and clarifications throughout the text, including many new and revised examples and explanations;
- new Microsoft® PowerPoint® files and Wolfram *Mathematica*® notebooks that coordinate with the text and that instructors may use in their teaching (see "Ancillary Materials" below).

How to Use This Book

There is more material in this book than can be covered comfortably during a single semester. Hence, the instructor will wish to eliminate some topics or subtopics—or to abbreviate the rather leisurely presentations of limits and differentiability. Since I frequently find myself without the time to treat surface integrals in detail, I have separated all material concerning parametrized surfaces, surface integrals, and Stokes's and Gauss's theorems (Chapter 7), from that concerning line integrals and Green's theorem (Chapter 6). In particular, in a one-semester course for students having little or no experience with vectors or matrices, instructors can probably expect to cover most of the material in Chapters 1–6, although no doubt it will be necessary to omit some of the optional subsections and to downplay many of the proofs of results. A rough outline for such a course, allowing for some instructor discretion, could be the following:

Chapter 1	8–9 lectures
Chapter 2	9 lectures
Chapter 3	4–5 lectures
Chapter 4	5–6 lectures
Chapter 5	8 lectures
Chapter 6	4 lectures
	38–41 lectures

If students have a richer background (so that much of the material in Chapter 1 can be left largely to them to read on their own), then it should be possible to treat a good portion of Chapter 7 as well. For a two-quarter or two-semester course, it should be possible to work through the entire book with reasonable care and rigor, although coverage of Chapter 8 should depend on students' exposure to introductory linear algebra, as somewhat more sophistication is assumed there.

The exercises vary from relatively routine computations to more challenging and provocative problems, generally (but not invariably) increasing in difficulty within each section. In a number of instances, groups of problems serve to introduce supplementary topics or new applications. Each chapter concludes with a set of miscellaneous exercises that both review and extend the ideas introduced in the chapter.

A word about the use of technology. The text was written without reference to any particular computer software or graphing calculator. Most of the exercises can be solved by hand, although there is no reason not to turn over some of the more tedious calculations to a computer. Those exercises that *require* a computer for computational or graphical purposes are marked with the symbol T and should be amenable to software such as *Mathematica*®, Maple®, or MATLAB.

Ancillary Materials

In addition to this text a **Student Solutions Manual** is available. An **Instructor's Solutions Manual**, containing complete solutions to all of the exercises, is available to course instructors from the Pearson Instructor Resource Center (www.pearsonhighered.com/irc), as are many Microsoft® PowerPoint® files and Wolfram *Mathematica*® notebooks that can be adapted for classroom use. The reader can find errata for the text and accompanying solutions manuals at the following address:
www.oberlin.edu/math/faculty/colley/VCErrata.html

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Finally, I thank the many Oberlin students who had the patience to listen to me lecture and who inspired me to write and improve this volume.

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To the Student: Some Preliminary Notation

Here are the ideas that you need to keep in mind as you read this book and learn vector calculus.

Given two sets A and B, I assume that you are familiar with the notation $A \cup B$ for the **union** of A and B—those elements that are in either A or B (or both):

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

Similarly, $A \cap B$ is used to denote the **intersection** of A and B—those elements that are in both A and B:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

The notation $A \subseteq B$, or $A \subset B$, indicates that A is a **subset** of B (possibly empty or equal to B).

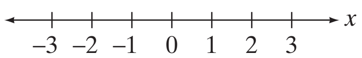

Figure 1 The coordinate line **R**.

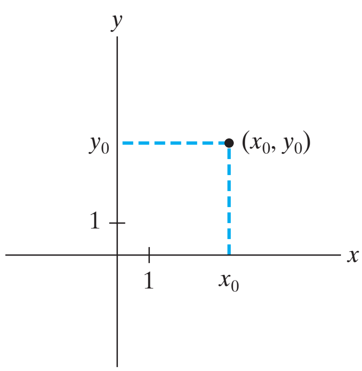

Figure 2 The coordinate plane $\mathbf{R}^2$.

One-dimensional space (also called the **real line** or **R**) is just a straight line. We put real number coordinates on this line by placing negative numbers on the left and positive numbers on the right. (See Figure 1.)

Two-dimensional space, denoted $\mathbf{R}^2$, is the familiar Cartesian plane. If we construct two perpendicular lines (the x- and y-**coordinate axes**), set the **origin** as the point of intersection of the axes, and establish numerical scales on these lines, then we may locate a point in $\mathbf{R}^2$ by giving an ordered pair of numbers (x, y), the **coordinates** of the point. Note that the coordinate axes divide the plane into four **quadrants**. (See Figure 2.)

Three-dimensional space, denoted $\mathbf{R}^3$, requires three mutually perpendicular coordinate axes (called the x-, y- and z-**axes**) that meet in a single point (called the **origin**) in order to locate an arbitrary point. Analogous to the case of $\mathbf{R}^2$, if we establish scales on the axes, then we can locate a point in $\mathbf{R}^3$ by giving an ordered triple of numbers (x, y, z). The coordinate axes divide three-dimensional space into eight **octants**. It takes some practice to get your sense of perspective correct when sketching points in $\mathbf{R}^3$. (See Figure 3.) Sometimes we draw the coordinate axes in $\mathbf{R}^3$ in different orientations in order to get a better view of things. However, we always maintain the axes in a **right-handed configuration**. This means that if you curl the fingers of your right hand from the positive x-axis to the positive y-axis, then your thumb will point along the positive z-axis. (See Figure 4.)

Although you need to recall particular techniques and methods from the calculus you have already learned, here are some of the more important concepts to keep in mind: Given a function $f(x)$, the **derivative** $f'(x)$ is the limit (if it exists) of the difference quotient of the function:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

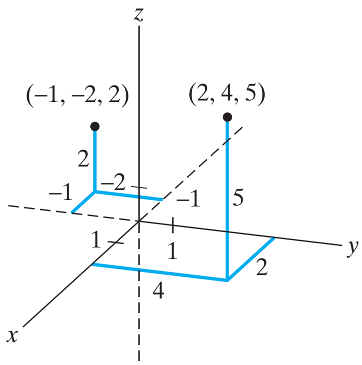

Figure 3 Three-dimensional space $\mathbf{R}^3$. Selected points are graphed.

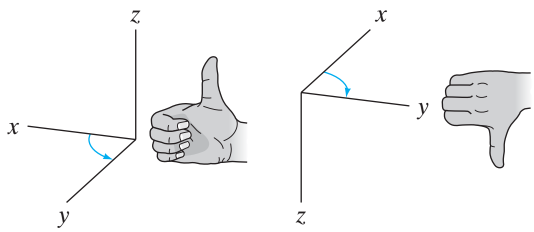

Figure 4 The x-, y-, and z-axes in $\mathbf{R}^3$ are always drawn in a right-handed configuration.

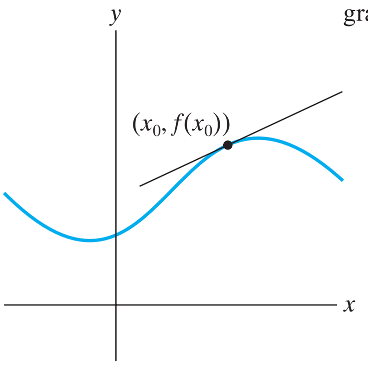

Figure 5 The derivative $f'(x_0)$ is the slope of the tangent line to $y = f(x)$ at $(x_0, f(x_0))$.

The significance of the derivative $f'(x_0)$ is that it measures the slope of the line tangent to the graph of f at the point $(x_0, f(x_0))$. (See Figure 5.) The derivative may also be considered to give the instantaneous rate of change of f at $x = x_0$. We also denote the derivative $f'(x)$ by df/dx.

The **definite integral** $\int_a^b f(x)\,dx$ of f on the closed interval $[a, b]$ is the limit (provided it exists) of the so-called **Riemann sums** of f:

$$\int_a^b f(x)\,dx = \lim_{\text{all } \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i.$$

Here $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ denotes a **partition** of $[a, b]$ into subintervals $[x_{i-1}, x_i]$, the symbol $\Delta x_i = x_i - x_{i-1}$ (the length of the subinterval), and x_i^* denotes any point in $[x_{i-1}, x_i]$. If $f(x) \geq 0$ on $[a, b]$, then each term $f(x_i^*)\Delta x_i$ in the Riemann sum is the area of a rectangle related to the graph of f. The Riemann sum $\sum_{i=1}^n f(x_i^*)\Delta x_i$ thus approximates the total area under the graph of f between $x = a$ and $x = b$. (See Figure 6.)

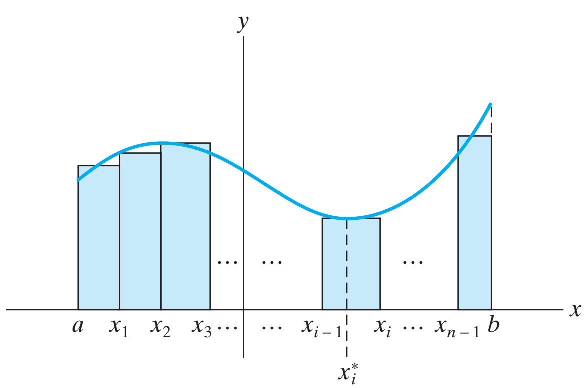

Figure 6 If $f(x) \geq 0$ on $[a, b]$, then the Riemann sum approximates the area under $y = f(x)$ by giving the sum of areas of rectangles.

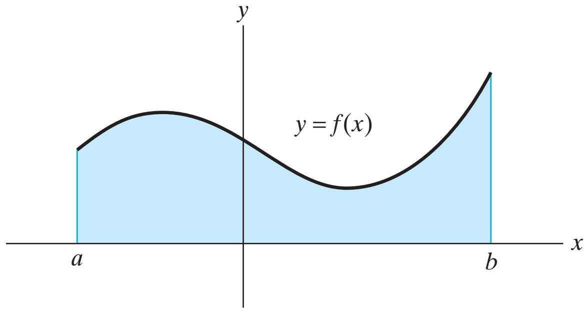

Figure 7 The area under the graph of $y = f(x)$ is $\int_a^b f(x)\,dx$.

The definite integral $\int_a^b f(x)\,dx$, if it exists, is taken to represent the area under $y = f(x)$ between $x = a$ and $x = b$. (See Figure 7.)

The derivative and the definite integral are connected by an elegant result known as **the fundamental theorem of calculus**. Let $f(x)$ be a continuous function of one variable, and let $F(x)$ be such that $F'(x) = f(x)$. (The function F is called an **antiderivative** of f.) Then

1. $$\int_a^b f(x)\,dx = F(b) - F(a);$$

2. $$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Finally, the end of an example is denoted by the symbol ◆ and the end of a proof by the symbol ■.

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Vector Calculus

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1 Vectors

1.1 Vectors in Two and Three Dimensions

For your study of the calculus of several variables, the notion of a vector is fundamental. As is the case for many of the concepts we shall explore, there are both *algebraic* and *geometric* points of view. You should become comfortable with both perspectives in order to solve problems effectively and to build on your basic understanding of the subject.

Vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$: The Algebraic Notion

DEFINITION 1.1 A **vector** in $\mathbf{R}^2$ is simply an ordered pair of real numbers. That is, a vector in $\mathbf{R}^2$ may be written as

$$(a_1, a_2) \quad \text{(e.g., } (1, 2) \text{ or } (\pi, 17)\text{)}.$$

Similarly, a **vector** in $\mathbf{R}^3$ is simply an ordered triple of real numbers. That is, a vector in $\mathbf{R}^3$ may be written as

$$(a_1, a_2, a_3) \quad \text{(e.g., } (\pi, e, \sqrt{2})\text{)}.$$

To emphasize that we want to consider the pair or triple of numbers as a single unit, we will use **boldface** letters; hence $\mathbf{a} = (a_1, a_2)$ or $\mathbf{a} = (a_1, a_2, a_3)$ will be our standard notation for vectors in $\mathbf{R}^2$ or $\mathbf{R}^3$. Whether we mean that $\mathbf{a}$ is a vector in $\mathbf{R}^2$ or in $\mathbf{R}^3$ will be clear from context (or else won't be important to the discussion). When doing handwritten work, it is difficult to "boldface" anything, so you'll want to put an arrow over the letter. Thus, $\vec{a}$ will mean the same thing as $\mathbf{a}$. Whatever notation you decide to use, it's important that you distinguish the *vector* $\mathbf{a}$ (or $\vec{a}$) from the *single real number* a. To contrast them with vectors, we will also refer to single real numbers as **scalars**.

In order to do anything interesting with vectors, it's necessary to develop some arithmetic operations for working with them. Before doing this, however, we need to know when two vectors are equal.

DEFINITION 1.2 Two vectors $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ in $\mathbf{R}^2$ are **equal** if their corresponding components are equal, that is, if $a_1 = b_1$ and $a_2 = b_2$. The same definition holds for vectors in $\mathbf{R}^3$: $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ are **equal** if their corresponding components are equal, that is, if $a_1 = b_1$, $a_2 = b_2$, and $a_3 = b_3$.

EXAMPLE 1 The vectors $\mathbf{a} = (1, 2)$ and $\mathbf{b} = \left(\frac{3}{3}, \frac{6}{3}\right)$ are equal in $\mathbf{R}^2$, but $\mathbf{c} = (1, 2, 3)$ and $\mathbf{d} = (2, 3, 1)$ are *not* equal in $\mathbf{R}^3$. ◆

Next, we discuss the operations of vector addition and scalar multiplication. We'll do this by considering vectors in $\mathbf{R}^3$ only; exactly the same remarks will hold for vectors in $\mathbf{R}^2$ if we simply ignore the last component.

DEFINITION 1.3 **(Vector Addition)** Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ be two vectors in $\mathbf{R}^3$. Then the **vector sum $\mathbf{a} + \mathbf{b}$** is the vector in $\mathbf{R}^3$ obtained via componentwise addition: $\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$.

EXAMPLE 2 We have $(0, 1, 3) + (7, -2, 10) = (7, -1, 13)$ and (in $\mathbf{R}^2$):

$$(1, 1) + (\pi, \sqrt{2}) = (1 + \pi, 1 + \sqrt{2}).$$ ◆

Properties of vector addition. We have

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ for all $\mathbf{a}, \mathbf{b}$ in $\mathbf{R}^3$ (commutativity);
2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$ for all $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in $\mathbf{R}^3$ (associativity);
3. a special vector, denoted $\mathbf{0}$ (and called the **zero vector**), with the property that $\mathbf{a} + \mathbf{0} = \mathbf{a}$ for all $\mathbf{a}$ in $\mathbf{R}^3$.

These three properties require proofs, which, like most facts involving the algebra of vectors, can be obtained by explicitly writing out the vector components. For example, for property 1, we have that if

$$\mathbf{a} = (a_1, a_2, a_3) \quad \text{and} \quad \mathbf{b} = (b_1, b_2, b_3),$$

then

$$\begin{aligned}\mathbf{a} + \mathbf{b} &= (a_1 + b_1, a_2 + b_2, a_3 + b_3)\\ &= (b_1 + a_1, b_2 + a_2, b_3 + a_3)\\ &= \mathbf{b} + \mathbf{a},\end{aligned}$$

since real number addition is commutative. For property 3, the "special vector" is just the vector whose components are all zero: $\mathbf{0} = (0, 0, 0)$. It's then easy to check that property 3 holds by writing out components. Similarly for property 2, so we leave the details as exercises.

DEFINITION 1.4 **(Scalar Multiplication)** Let $\mathbf{a} = (a_1, a_2, a_3)$ be a vector in $\mathbf{R}^3$ and let $k \in \mathbf{R}$ be a scalar (real number). Then the **scalar product $k\mathbf{a}$** is the vector in $\mathbf{R}^3$ given by multiplying each component of $\mathbf{a}$ by k: $k\mathbf{a} = (ka_1, ka_2, ka_3)$.

EXAMPLE 3 If $\mathbf{a} = (2, 0, \sqrt{2})$ and $k = 7$, then $k\mathbf{a} = (14, 0, 7\sqrt{2})$. ◆

The results that follow are not difficult to check—just write out the vector components.

Properties of scalar multiplication. For all vectors **a** and **b** in $\mathbf{R}^3$ (or $\mathbf{R}^2$) and scalars k and l in $\mathbf{R}$, we have

1. $(k + l)\mathbf{a} = k\mathbf{a} + l\mathbf{a}$ (distributivity);
2. $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$ (distributivity);
3. $k(l\mathbf{a}) = (kl)\mathbf{a} = l(k\mathbf{a})$.

It is worth remarking that none of these definitions or properties really depends on dimension, that is, on the number of components. Therefore we could have introduced the algebraic concept of a vector in $\mathbf{R}^n$ as an **ordered *n*-tuple** $(a_1, a_2, \ldots, a_n)$ of real numbers and defined addition and scalar multiplication in a way analogous to what we did for $\mathbf{R}^2$ and $\mathbf{R}^3$. Think about what such a generalization means. We will discuss some of the technicalities involved in §1.6.

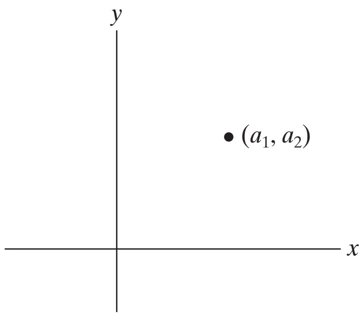

Figure 1.1 A vector $\mathbf{a} \in \mathbf{R}^2$ corresponds to a point in $\mathbf{R}^2$.

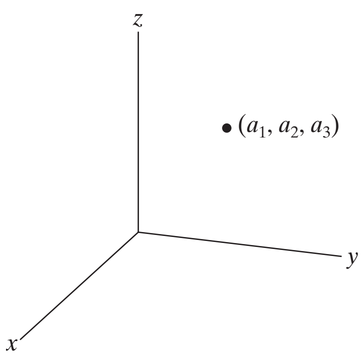

Figure 1.2 A vector $\mathbf{a} \in \mathbf{R}^3$ corresponds to a point in $\mathbf{R}^3$.

Vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$: The Geometric Notion

Although the algebra of vectors is certainly important and you should become adept at working algebraically, the formal definitions and properties tend to present a rather sterile picture of vectors. A better motivation for the definitions just given comes from geometry. We explore this geometry now. First of all, the fact that a vector **a** in $\mathbf{R}^2$ is a pair of real numbers (a_1, a_2) should make you think of the coordinates of a point in $\mathbf{R}^2$. (See Figure 1.1.) Similarly, if $\mathbf{a} \in \mathbf{R}^3$, then **a** may be written as (a_1, a_2, a_3), and this triple of numbers may be thought of as the coordinates of a point in $\mathbf{R}^3$. (See Figure 1.2.)

All of this is fine, but the results of performing vector addition or scalar multiplication don't have very interesting or meaningful geometric interpretations in terms of points. As we shall see, it is better to visualize a vector in $\mathbf{R}^2$ or $\mathbf{R}^3$ as an arrow that begins at the origin and ends at the point. (See Figure 1.3.) Such a depiction is often referred to as the **position vector** of the point (a_1, a_2) or (a_1, a_2, a_3).

If you've studied vectors in physics, you have heard them described as objects having "magnitude and direction." Figure 1.3 demonstrates this concept, provided that we take "magnitude" to mean "length of the arrow" and "direction" to be the orientation or sense of the arrow. (Note: There is an exception to this approach, namely, the zero vector. The zero vector just sits at the origin, like a point, and has no magnitude and, therefore, an indeterminate direction. This exception will not pose much difficulty.) However, in physics, one doesn't demand that all vectors

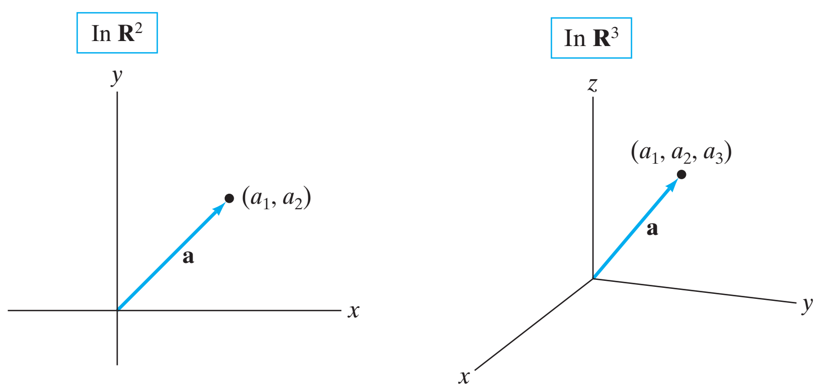

Figure 1.3 A vector **a** in $\mathbf{R}^2$ or $\mathbf{R}^3$ is represented by an arrow from the origin to **a**.

be represented by arrows having their tails bound to the origin. One is free to "parallel translate" vectors throughout $\mathbf{R}^2$ and $\mathbf{R}^3$. That is, one may represent the vector $\mathbf{a} = (a_1, a_2, a_3)$ by an arrow with its tail at the origin (and its head at (a_1, a_2, a_3)) or with its tail at any other point, so long as the length and sense of the arrow are not disturbed. (See Figure 1.4.) For example, if we wish to represent $\mathbf{a}$ by an arrow with its tail at the point (x_1, x_2, x_3), then the head of the arrow would be at the point $(x_1 + a_1, x_2 + a_2, x_3 + a_3)$. (See Figure 1.5.)

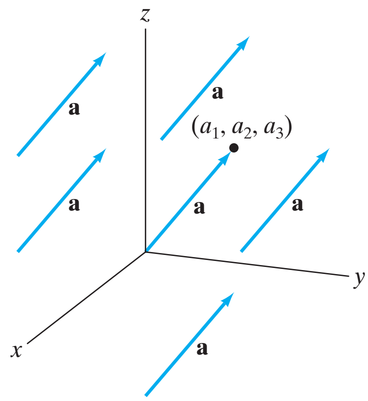

Figure 1.4 Each arrow is a parallel translate of the position vector of the point (a_1, a_2, a_3) and represents the same vector.

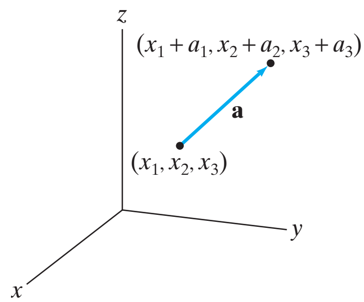

Figure 1.5 The vector $\mathbf{a} = (a_1, a_2, a_3)$ represented by an arrow with tail at the point (x_1, x_2, x_3).

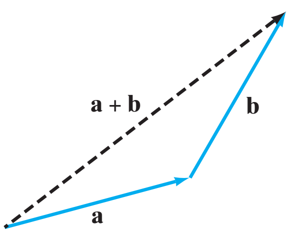

Figure 1.6 The vector $\mathbf{a} + \mathbf{b}$ may be represented by an arrow whose tail is at the tail of $\mathbf{a}$ and whose head is at the head of $\mathbf{b}$.

With this geometric description of vectors, vector addition can be visualized in two ways. The first is often referred to as the "head-to-tail" method for adding vectors. Draw the two vectors $\mathbf{a}$ and $\mathbf{b}$ to be added so that the tail of one of the vectors, say $\mathbf{b}$, is at the head of the other. Then the vector sum $\mathbf{a} + \mathbf{b}$ may be represented by an arrow whose tail is at the tail of $\mathbf{a}$ and whose head is at the head of $\mathbf{b}$. (See Figure 1.6.) Note that it is *not* immediately obvious that $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ from this construction!

The second way to visualize vector addition is according to the so-called **parallelogram law**: If $\mathbf{a}$ and $\mathbf{b}$ are nonparallel vectors drawn with their tails emanating from the same point, then $\mathbf{a} + \mathbf{b}$ may be represented by the arrow (with its tail at the common initial point of $\mathbf{a}$ and $\mathbf{b}$) that runs along a diagonal of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$ (Figure 1.7). The parallelogram law is completely consistent with the head-to-tail method. To see why, just parallel translate $\mathbf{b}$ to the opposite side of the parallelogram. Then the diagonal just described is the result of adding $\mathbf{a}$ and (the translate of) $\mathbf{b}$, using the head-to-tail method. (See Figure 1.8.)

We still should check that these geometric constructions agree with our algebraic definition. For simplicity, we'll work in $\mathbf{R}^2$. Let $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ as usual. Then the arrow obtained from the parallelogram law addition of $\mathbf{a}$ and $\mathbf{b}$ is the one whose tail is at the origin O and whose head is at the point P in Figure 1.9. If we parallel translate $\mathbf{b}$ so that its tail is at the head of $\mathbf{a}$, then it is immediate that the coordinates of P must be $(a_1 + b_1, a_2 + b_2)$, as desired.

Scalar multiplication is easier to visualize: The vector $k\mathbf{a}$ may be represented by an arrow whose length is $|k|$ times the length of $\mathbf{a}$ and whose direction is the same as that of $\mathbf{a}$ when $k > 0$ and the opposite when $k < 0$. (See Figure 1.10.)

It is now a simple matter to obtain a geometric depiction of the **difference** between two vectors. (See Figure 1.11.) The difference $\mathbf{a} - \mathbf{b}$ is nothing more

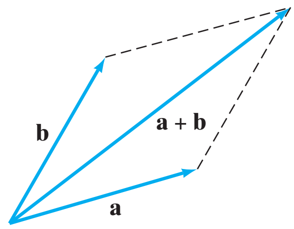

Figure 1.7 The vector $\mathbf{a}+\mathbf{b}$ may be represented by the arrow that runs along the diagonal of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.

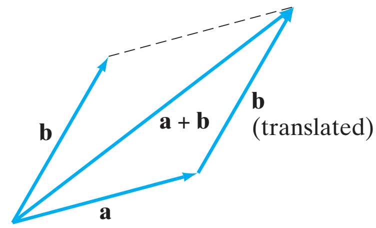

Figure 1.8 The equivalence of the parallelogram law and the head-to-tail methods of vector addition.

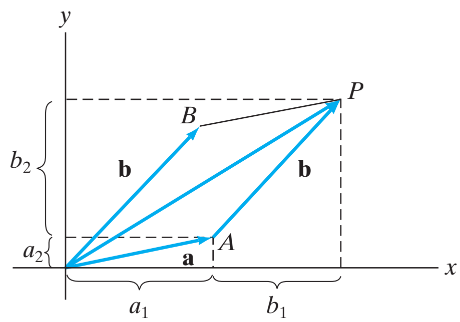

Figure 1.9 The point P has coordinates (a_1+b_1, a_2+b_2).

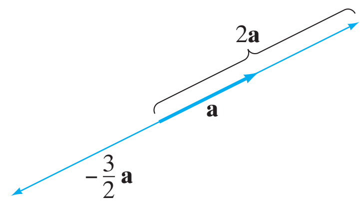

Figure 1.10 Visualization of scalar multiplication.

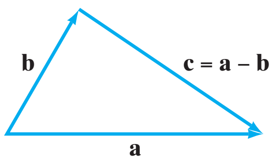

Figure 1.11 The geometry of vector subtraction. The vector $\mathbf{c}$ is such that $\mathbf{b}+\mathbf{c}=\mathbf{a}$. Hence, $\mathbf{c}=\mathbf{a}-\mathbf{b}$.

than $\mathbf{a}+(-\mathbf{b})$ (where $-\mathbf{b}$ means the scalar -1 times the vector $\mathbf{b}$). The vector $\mathbf{a}-\mathbf{b}$ may be represented by an arrow pointing from the head of $\mathbf{b}$ toward the head of $\mathbf{a}$; such an arrow is also a diagonal of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$. (As we have seen, the other diagonal can be used to represent $\mathbf{a}+\mathbf{b}$.)

Here is a construction that will be useful to us from time to time.

DEFINITION 1.5 Given two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ in $\mathbf{R}^3$, the **displacement vector from P_1 to P_2** is

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1).$$

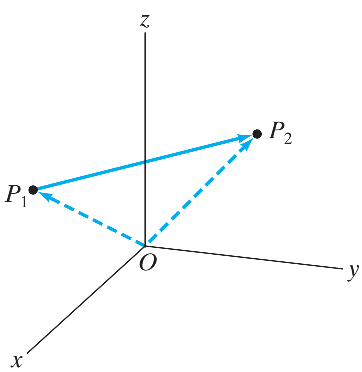

Figure 1.12 The displacement vector $\overrightarrow{P_1P_2}$, represented by the arrow from P_1 to P_2, is the difference between the position vectors of these two points.

This construction is not hard to understand if we consider Figure 1.12. Given the points P_1 and P_2, draw the corresponding position vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$. Then we see that $\overrightarrow{P_1P_2}$ is precisely $\overrightarrow{OP_2}-\overrightarrow{OP_1}$. An analogous definition may be made for $\mathbf{R}^2$.

In your study of the calculus of one variable, you no doubt used the notions of derivatives and integrals to look at such physical concepts as velocity, acceleration, force, etc. The main drawback of the work you did was that the techniques involved allowed you to study only *rectilinear,* or straight-line, activity. Intuitively, we all understand that motion in the plane or in space is more complicated than straight-line motion. Because vectors possess direction as well as magnitude, they are ideally suited for two- and three-dimensional dynamical problems.